**ECONOMICS 7360**

**PROBLEM SET #1**

**1.** Consider the optimal commodity taxation model we used to derive the *Many Person Ramsey Tax Rule*. Assume that there are just two consumer goods (i.e., $n = 2$) and that all the consumers are identical. In the notes, it was asserted that it could be shown that, under these assumptions, the optimal tax rates satisfy the equation

$$t_1 - t_2 = -\frac{1-\overline{b}}{D} x_1 x_2 [\varepsilon_{1l} - \varepsilon_{2l}],$$

where all the variables are defined in the notes. Prove that this is the case. (Hint: when you get stuck, try consulting Diamond and Mirrlees "Optimal Taxation and Public Production II: Tax Rules" *American Economic Review*, 1971).

**2.** Consider a version of the Mirrlees Model considered in class with just two ability types: *high types* with ability $a_H$ and *low types* with ability $a_L$, where $a_H > a_L > 0$. Let $\lambda$ be the fraction of low types in the population.

**a.** An *allocation* in this economy is described by $\{(x_L, y_L), (x_H, y_H)\}$. To be *feasible* an allocation must satisfy the aggregate resource constraint

$$\lambda x_L + (1-\lambda) x_H + G \leq \lambda y_L + (1-\lambda) y_H.$$

The utility of an individual with high ability under such an allocation is $x_H - \varphi(y_H/a_H)$ while that of an individual with low ability is $x_L - \varphi(y_L/a_L)$. An allocation is *Pareto efficient* if, for some $\underline{u}$, it solves the problem

$$\begin{array}{c} \max_{\{(x_L,y_L),(x_H,y_H)\}} x_H - \varphi(y_H/a_H) \\ s.t.\ x_L - \varphi(y_L/a_L) \geq \underline{u} \quad (U_L) \\ \lambda x_L + (1-\lambda) x_H + G \leq \lambda y_L + (1-\lambda) y_H. \quad (R) \end{array} \tag{1}$$

Show that an allocation (with positive consumptions - $x_L, x_H > 0$) is Pareto efficient if and only if the earnings pair $(y_L, y_H)$ maximizes Marshallian aggregate surplus

$$\lambda(y_L - \varphi(y_L/a_L)) + (1-\lambda)(y_H - \varphi(y_H/a_H))$$

and the consumption pair $(x_L, x_H)$ is such that $U_L$ and $R$ hold with equality. Then use first order conditions to characterize the surplus maximizing earnings pair. Interpret what the conditions are telling you.

**b.** Show that if the government can observe individuals' abilities, it can implement any Pareto efficient utility allocation through a simple tax system.

**c.** Assume that the government cannot observe individuals' abilities but can observe their incomes. Suppose the government wants to implement some Pareto efficient allocation

$\{(x_L, y_L), (x_H, y_H)\}$ and it just announces that individuals who earn income level $y_H$ must pay taxes $y_H - x_H$ while those who earn income level $y_L$ must pay taxes $y_L - x_L$. Will this implement the allocation? Explain carefully.

**d.** Consider the *second best* efficiency problem in which problem (1) is amended to include the following incentive constraints:

$$x_H - \varphi(y_H/a_H) \geq x_L - \varphi(y_L/a_H) \quad (IC_H)$$

and

$$x_L - \varphi(y_L/a_L) \geq x_H - \varphi(y_H/a_L) \quad (IC_L).$$

To make the problem interesting, assume that $G$ and $\underline{u}$ are sufficiently high that the Pareto efficient allocation in which those who are low types have utility $\underline{u}$ violates the high types' incentive constraint. Prove the following result: if $\{(x_L, y_L), (x_H, y_H)\}$ satisfies $(IC_H)$ with equality and is such that $y_L < y_H$. Then it satisfies $(IC_L)$. (Hint: You might find a diagram useful.)

**e.** Your answer to part d justifies considering the following "relaxed problem"

$$\max_{\{(x_L,y_L),(x_H,y_H)\}} x_H - \varphi(y_H/a_H)$$
$$s.t.\ (U_L),\ (R),\ \&\ (IC_H)$$

where we have thrown away the low type's incentive constraint. Provided the solution to this relaxed problem is such that $(IC_H)$ holds with equality and $y_L < y_H$, it will satisfy both incentive constraints and hence solve the "unrelaxed problem". Show that if $\{(x_L, y_L), (x_H, y_H)\}$ solves the relaxed problem, then it satisfies $(U_L)$ & $(IC_H)$ with equality.

**f.** Use your answer to part e to solve out for $x_L$ and $x_H$ as functions of $y_L$ and $y_H$, thereby reducing the relaxed problem to an unconstrained problem involving the choice variables $y_L$ and $y_H$. Characterize the second best efficient earnings levels with first order conditions and show that $y_L < y_H$. Interpret your first order conditions and explain how they differ from those that characterize the Pareto efficient earnings levels.

**g.** What does your answer to part f imply about optimal marginal tax rates? Explain carefully.

**PROBLEM SET #2**

**1.** Prove Proposition 1 in the Section on the Neo-classical Theory of Public Goods.

**2.** Consider the two ability type Mirrlees Model you analyzed in Problem Set #1. Extend the model to include a non-excludable public good in the following way. Let $g$ denote the level of the public good and assume individuals have utility functions

$$x + B(g) - \varphi(l),$$

where as before $x$ denotes private good consumption and $l$ denotes labor. The function $B(g)$ is increasing, smooth and strictly concave and is such that $B(0) = 0$. The cost of the public good is $c$ per unit, where $c < B'(0)$. The public good is provided by the government and it has no other revenue needs (i.e., it just needs to raise revenue of $cg$).

Consider the second best efficiency problem for this economy and prove that the optimal level of the public good satisfies the Samuelson condition (i.e., $B'(g) = c$). (You may assume that the constraints that each type's private consumption be positive are satisfied).

**3.** Consider the following dynamic public good model. There are $n$ infinitely-lived consumers indexed by $i = 1, ..., n$. Periods are indexed by $t = 0, 1, 2, ....$ There are two goods - a numeraire private good $z$ and a durable public good $x$. Each consumer $i$ has per period utility function

$$z_i + \varphi(x),$$

where $\varphi(\cdot)$ is increasing, smooth, and strictly concave. In each period, each consumer obtains an income $y$. The cost (in terms of units of the numeraire) of providing a unit of public good is $c$. Consumers have discount rates $\beta \in (0, 1)$. The level of the public good evolves through time according to the equation

$$x_{t+1} = (1 - \delta)x_t + I_t,$$

where $I_t \geq 0$ denotes investment in the public good in period $t$ and $\delta \in (0, 1)$ denotes the depreciation rate. Investment in period $t$ is financed by a uniform tax of $cI_t/n$ levied on each consumer.

Assuming that the community begins with zero public good (i.e., $x_0 = 0$) and that $B'(0) > c(1 - \beta(1 - \delta))$, characterize the investment path that maximizes the consumers' utility. (You may assume that the constraints that each consumer's private consumption be positive are satisfied).

## PROBLEM SET #3

**1.** Show that the consumer-consumer and producer-consumer externality examples discussed in class, fit into the framework of Weitzman's model. More specifically, describe what $q$ would correspond to in these examples and explicitly compute what the benefit and cost functions $B(q)$ and $C(q)$ would be.

**2.** In the tradeable permit model developed in class with identical firms, show that the government can achieve efficiency by either imposing a tax $t = nD'(\gamma(a^e)X^e)$ on emissions or providing each firm with a permit to emit at most $\rho = \gamma(a^e)X^e/m$ units of pollution.

**3.** In the tradeable permit model developed in class with two types of firms, show that the government can achieve efficiency by providing each firm with a permit to emit at most $\rho = \gamma(a^e)X^e/m$ units of pollution and allowing firms to trade permits. (Hint: To allow tradeable permits, you will need to introduce a market for permits along with the market for the product. Thus, there will be a price for permits, say $r$, and a price for the product and given these prices firms will choose how many permits to buy or sell and how much to produce. In equilibrium, permit supply must equal permit demand and product supply must equal product demand. To answer the question, you only need to describe an equilibrium in which efficiency results rather than characterizing the entire set of equilibria. I suggest focusing on the equilibrium in which all the firms who can abate produce the same output level and hence buy the same number of permits.)

**PROBLEM SET #4**

**1.** Consider the following partial equilibrium model for looking at the design of cash assistance programs. There is a population of potentially poor people (e.g., single parents) divided into two types according to their income generating ability or wage, denoted by $a_i$, where $0 < a_1 < a_2$. Let $\pi_i$ denote the fraction of type $i$'s. Each individual is endowed with $T$ units of time and has a quasi-linear utility function

$$u = x - \frac{l^{(1+\frac{1}{\varepsilon})}}{1+\varepsilon}$$

where $x$ is consumption and $l$ is labor supply. In the absence of government intervention, individuals of type $i$ would have incomes

$$\widetilde{y}_i = \arg\max\{y - \frac{\left(\frac{y}{a_i}\right)^{(1+\frac{1}{\varepsilon})}}{1+\varepsilon}\}$$

and enjoy utility levels

$$\widetilde{u}_i = \widetilde{y}_i - \frac{\left(\frac{\widetilde{y}_i}{a_i}\right)^{(1+\frac{1}{\varepsilon})}}{1+\varepsilon}.$$

The government is concerned that all citizens utility be above some target minimal level $\underline{u}$ but desires to ensure that this objective is reached at minimum fiscal cost. Assume that

$$\widetilde{u}_2 > \underline{u} > \widetilde{u}_1,$$

and that the government cannot observe individuals' abilities.

**(a)** Suppose the government is thinking of employing a benefit schedule of the form

$$B(y) = \begin{cases} G - ry & \text{if } y \leq G/r \\ 0 & \text{if } y > G/r \end{cases},$$

where parameters $G$ and $r$ are to be determined. Set up the government's optimization problem and characterize the optimal levels of $G$ and $r$.

**(b)** Characterize the optimal non-linear benefit schedule following the approach you employed to study the two type Mirrlees model in Problem Set #1.

**(c)** Suppose the government can impose public work requirements on benefit recipients. Would it be a good idea to require individuals to do public work for their benefits? (Hint: To study this, think of the government as offering a pair of consumption-income-public work bundles $(x_i, y_i, c_i)_{i=1}^2$, one intended for type 1 individuals and the other for type 2 individuals. The interpretation is that an individual with earnings $y_i$ receives benefits $x_i - y_i$ in exchange for $c_i$ units of public work. You want to explore whether the optimal $c_i$'s are positive or zero. Assume that public work is unproductive.)

**2.** Prove Proposition 3 from the notes on the New Dynamic Public Finance.

**3.** Consider the following numerical version of Coate's Samaritan's Dilemma model. The poor person has utility function $\ln x_p$ and each rich person has utility $x_r^i + 2\ln x_p$ $(i = 1, 2)$. Suppose that $y_p = 3$, $L = 5$, $\pi = 0.2$ and $y_r = 20$.

**(a)** Assuming that the government seeks to maximize the aggregate utility of the rich, solve for the optimal government transfer to the poor person under the assumption that the rich can commit not to bail him out in the event of an uninsured loss.

**(b)** Suppose that the government provides the optimal transfer from part (a) but that the rich cannot commit not to bail out the poor person in the event of an uninsured loss. Solve for the charitable transfer that the poor person would receive from the rich if he did not take out insurance and experienced a loss.

**(c)** Suppose that the government provides the optimal transfer from part (a) and the rich cannot commit. Would the poor person take out insurance or not?

**(d)** If the poor person does not take out insurance, show that the allocation of resources is Pareto inefficient. (Hint: The easiest way to do this is to find a Pareto dominating allocation.)